

Lecture 9 — February 9, 2021

Prof. Emmanuel Candes

Scribe: Carlos A. Sing-Long, Edited by E. Bates

1 Outline

Agenda: Finishing up DFTs, then shifting to applications!

1. X-ray propagation and Beer's Law
2. Motivation for X-ray tomography
3. Radon transform

Last Time: We discussed quadrature rules, which are means of estimating integrals of functions using a weighted sum of function values. Among these, one of the simplest is the trapezoidal rule, which derives its name from the fact that it attempts to approximate the area under the curve using a trapezoid. Although it can be proved the accuracy of this rule increases quadratically with the number of quadrature points, in some cases the accuracy increases exponentially. We showed how this phenomenon can be understood in terms of the Fourier transform, as the accuracy can be related to the rate of decay of the Fourier coefficients of the function. For a smooth and periodic function these decay faster than any polynomial, which in turn explains the spectral accuracy of the trapezoidal rule for this class of functions.

We also introduced the Fourier transform for finite signals, which is called the discrete Fourier transform (DFT). We showed its properties are analogue to those of the Fourier transform and the Fourier series. In particular, by defining the circular convolution of two finite signals and the circular shift, we showed that every time-invariant map acting on finite signals is given by a circular convolution by a fixed function. We then discussed and analyzed the fast Fourier transform (FFT) which is the *de facto* algorithm used to compute the discrete Fourier transform. This is due to the fact its computational complexity is almost linear in the length of the signal (modulo a logarithmic factor).

2 Medical imaging

One of the main advances in medicine in the last decades has been the development of **medical imaging**. These techniques allow to capture images of the human body by non-invasive means. In other words, it is not necessary to perform a surgical intervention to observe the human body and perform a diagnosis. The contributions made in this field have been recognized by prestigious awards

such as the Nobel Prize in Physiology or Medicine in 1979 to Allan M. Cormack and Godfrey N. Hounsfield “*for the development of computer assisted tomography,*” and in 2003 to Peter Mansfield and Paul Lauterbur “*for their discoveries concerning magnetic resonance imaging.*”

Broadly speaking, medical imaging techniques observe the outcome of the interaction between the human body and a known energy source. From this information, they attempt to recover a physical quantity of interest. We will study two of the most common imaging techniques: **computerized tomography (CT)**, which measures the attenuation that X-rays experience as they traverse the human body from multiple directions, and **magnetic resonance imaging (MRI)**, which measures the response of the water molecules in the human body when subjected to electromagnetic pulses. Surprisingly, both of these techniques are closely related to the Fourier transform.

We begin by discussing computerized tomography.

3 X-ray propagation and Beer’s law

An **X-ray** is a flux of high-energy electromagnetic radiation. Due to **de Broglie’s equations**, we have

$$E = \underbrace{h}_{\text{Planck's constant}} \times \underbrace{\nu}_{\text{frequency}},$$

and consequently the X-ray is equivalently characterized either by its high frequency. Because of this, we can use a **geometric wave propagation** approximation and model it as a vector field $\mathbf{I} = I(\mathbf{x})$ for $\mathbf{x} \in \mathbb{R}^2$ or $\mathbf{x} \in \mathbb{R}^3$, where

1. $\|I(\mathbf{x})\|$ indicates the intensity of the flux at a location $\mathbf{x}$,
2. $I(\mathbf{x})/\|I(\mathbf{x})\|$ indicates the direction of the flux at location $\mathbf{x}$.

In medical imaging we observe the changes X-rays experience after interacting with matter. A typical setup can be seen in Fig. 1. To be more precise, we can compare some physical property of the X-rays, such as energy and direction of propagation, before and after some interaction with matter. We hope this comparison will help us understand and recover the structure of the object that interacted with the X-rays. The task at hand is an example of an **inverse problem**, “inverse” in the sense that we are trying to reconstruct the cause after observing the effect.

To do so, though, we need a **forward model**, that is, a description of the interaction between X-rays and matter (i.e. establishing how we believe matter causes effects in X-rays). In particular, this model should characterize how the above physical properties change as the X-ray traverses matter. The expressions thus obtained will be the starting point for the mathematical study of tomography: solving our inverse problem.

Our model shall make the following assumptions:

- (A1) **No refraction nor diffraction:** The X-rays travel along straight lines that do not bend when interacting with an object.
- (A2) **Monochromatic rays:** The X-rays all have the same frequency.

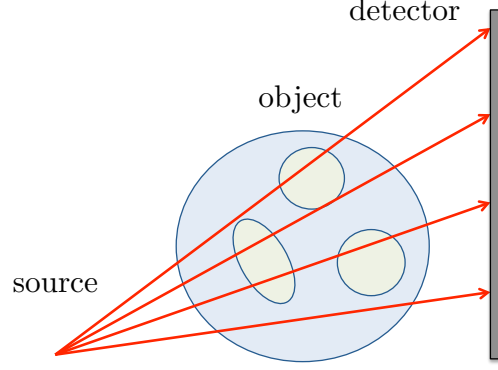


Figure 1: A generic setup for X-ray tomography. A source emits X-rays in many directions. These traverse matter and as a consequence they are attenuated and possibly scattered. A detector typically measures the energy of the incident rays after they have interacted with an object.

- (A3) **Beer's law:** Each material has a characteristic linear attenuation coefficient μ for rays emitted at a given frequency. More precisely,

$$\frac{dI}{ds} = -\mu(x(s))I(s),$$

where s is the arc length along the trajectory described by the ray.

These assumptions hold reasonably well in cases of interest:

- (A1) This is a very good approximation as X-rays have short wavelength. making the refraction and diffraction effects negligible.
- (A2) This is not always true, but it is a good approximation. Furthermore, it is useful for formulating a linear model.
- (A3) From (A2) we can write the vector field that models the X-rays as

$$I(\mathbf{x}) = EN(\mathbf{x}),$$

where $\mathbf{N} = N(\mathbf{x})$ is the **photon flux** and E is simply the energy of the photons. From (A2) we can assume the vector field $\mathbf{I}$ that models the X-rays correspond to straight lines. For the energies typically used for X-rays (5-10 keV) the change in the number of photons between two points $\mathbf{x}(s)$ and $\mathbf{x}(s + \Delta s)$ along the ray for $\Delta s \ll 1$ is explained mostly by **photon absorption**. Since $\mathbf{x}(s)$ parameterizes a straight line, this can be modeled as

$$N(\mathbf{x}(s + \Delta s)) - N(\mathbf{x}(s)) \approx -\mu(\mathbf{x}(s))N(\mathbf{x}(s))\Delta s \implies I(\mathbf{x}(s + \Delta s)) - I(\mathbf{x}(s)) \approx -\mu(\mathbf{x}(s))\Delta s,$$

which is simply a statistical model for the absorption. Here the rate of absorption is proportional to the number of photons. This is precisely Beer's law.

Under these hypotheses we can find an explicit form for Beer's law. We can parameterize a ray with source¹ at $\mathbf{x}_0$ as

$$\mathbf{x}(s, \mathbf{x}_0) = \mathbf{x}_0 + s\mathbf{v},$$

¹Here we use the term a bit loosely since in some cases the source will be assumed to be at infinity.

where $s > 0$, and $\mathbf{v}$ ($\|\mathbf{v}\| = 1$) is the direction of propagation. For an X-ray with source at $\mathbf{x}_0$, we have

$$\frac{dI_{\mathbf{x}_0}}{ds} = -\mu(\mathbf{x}(s))I_{\mathbf{x}_0}(s) \quad \implies \quad I_{\mathbf{x}_0}(s) = I_{\mathbf{x}_0}(0)e^{-\int_0^s \mu(\mathbf{x}_0 + \tau\mathbf{v}) d\tau}.$$

By measuring the intensity at two points $\mathbf{x}_{\text{in}} = \mathbf{x}_0 + s_{\text{in}}\mathbf{v}$ and $\mathbf{x}_{\text{out}} = \mathbf{x}_0 + s_{\text{out}}\mathbf{v}$, we deduce

$$\log\left(\frac{I_{\mathbf{x}_0}(\mathbf{x}_{\text{out}})}{I_{\mathbf{x}_0}(\mathbf{x}_{\text{in}})}\right) = -\int_{s_{\text{in}}}^{s_{\text{out}}} \mu(\mathbf{x}_0 + \tau\mathbf{v}) d\tau.$$

From this expression, we see that the change in the energy along a ray encodes information about the coefficient of absorption of the object. However, does the attenuation depend *only* on the object? To answer this question, we need to consider the effect of **beam spreading**. It is a well-known fact that when X-rays are emitted from a source placed at the origin,

$$E_0 = \int_{\Sigma} \langle I(\mathbf{x}), \boldsymbol{\nu}(\mathbf{x}) \rangle dS(\mathbf{x}),$$

for any closed surface Σ containing the origin, where $\boldsymbol{\nu}$ is the outward-pointing unit normal to Σ , and dS is the surface element. In other words, the irradiated energy E_0 must remain constant. By specializing this to a family of balls centered at the origin, and assuming the source is **isotropic** ($I(\mathbf{x}) = I(\|\mathbf{x}\|)$ is rotationally invariant), then

$$E_0 = \int_{B(0,r)} \langle I(\mathbf{x}), \boldsymbol{\nu}(\mathbf{x}) \rangle dS(\mathbf{x}) = c_d r^{d-1} \|I(r)\|,$$

where d is the dimension of the space ($d = 2$ or $d = 3$), and c_d is the area of the boundary of the unit ball in $\mathbb{R}^d$. Consequently,

$$\|I(r)\| = \frac{E_0}{c_d} \frac{1}{r^{d-1}},$$

and there is a natural decay of the intensity of the ray due to spreading. Note this is similar to Beer's law, and it implies the change in intensity will be a consequence of *both* absorption and spreading. To avoid the complications arising from this, we will make the **non-diverging source** assumption, that is, that the intensity of the X-ray does not decay due to spreading. In this case, attenuation is explained *solely* by interaction with matter.

4 X-ray tomography

The model formulated in the previous section shows that the interaction between an X-ray and an object is characterized by the absorption coefficient μ , and we can characterize the object by recovering this function. Here we specialize our problem to $d = 2$ and we $\mathbf{e}_1$ and $\mathbf{e}_2$ denote the canonical basis vectors. Using the experimental setup on Fig. 2a, we see that $I_{\mathbf{x}_0}(0) = I_0$ and Beer's law states that

$$\log\left(\frac{I_{\mathbf{x}_0}(D)}{I_{\mathbf{x}_0}(0)}\right) = \log\left(\frac{I_{\text{out}}(t)}{I_0}\right) = -\int_0^D \mu(t\mathbf{e}_1 + (D - \tau)\mathbf{e}_2) d\tau.$$

Can we recover μ from I_{out} ? The answer is clearly *no*. The main issue here is the non-injectivity of the above map, which is a consequence of the loss of depth information (see Fig. 2b).

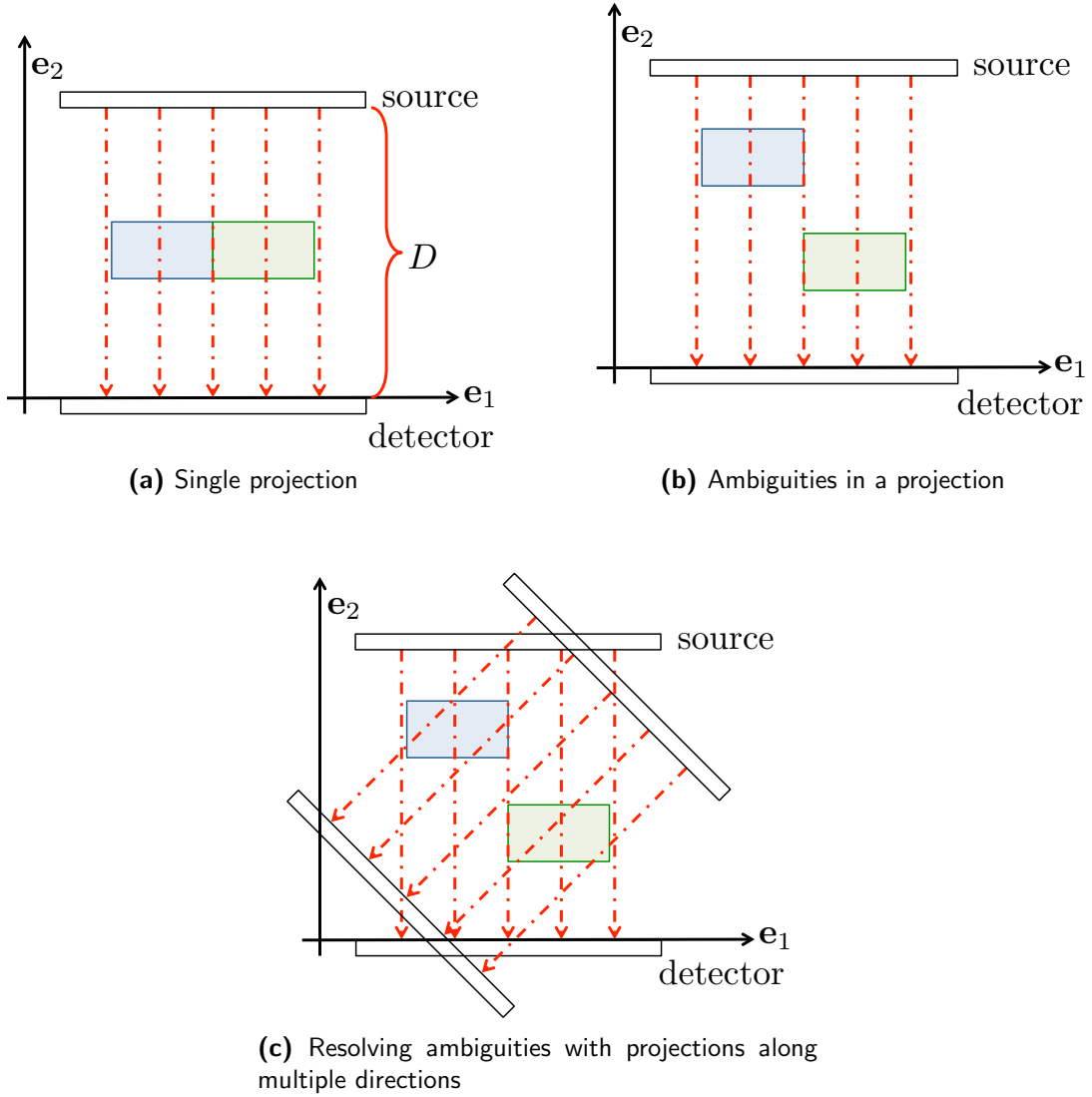


Figure 2: Irradiating an object from a single direction can cause ambiguities, as two very different objects may produce the same measurements. However, we can try to resolve these ambiguities by rotating the source and detector.

However, one could think about measuring more and more projections of the object to resolve these ambiguities (see Fig. 2c). Intuitively, with more projections one should be able to characterize the original object more accurately. Thus, can we recover μ from line integrals taken from all possible directions? And, if this is the case, is there a reconstruction formula? We will soon see that the answer to both questions is *yes*.

5 Radon transform

To answer the questions posed last time, we shall develop the machinery needed to handle the projection information given. In particular, for $f \in L^2(\mathbb{R}^2)$, define the **Radon transform** of f as

the function $\mathcal{R}f : \mathbb{R} \times [0, \pi) \mapsto \mathbb{C}$ given by

$$\mathcal{R}f(t, \theta) = \int_{L_{t, \theta}} f(\mathbf{x}) dS(\mathbf{x}) = \iint f(\mathbf{x}) \delta(\langle \mathbf{x}, \mathbf{n}_\theta \rangle - t) d\mathbf{x}, \quad (1)$$

where $\mathbf{n}_\theta = (\cos \theta, \sin \theta)$ is the unit vector with angle θ . $\mathcal{R}f(t, \theta)$ is the integral of f along the line $L_{t, \theta}$ whose direction is perpendicular to $\mathbf{n}_\theta$ and whose distance from the origin is t :

$$L_{t, \theta} = \{\mathbf{x} : \langle \mathbf{x}, \mathbf{n}_\theta \rangle = t\}.$$

Indeed, using the notation $\mathbf{n}_\theta^\perp = (-\sin \theta, \cos \theta)$, we can alternatively write

$$\mathcal{R}f(t, \theta) = \int f(t\mathbf{n}_\theta + s\mathbf{n}_\theta^\perp) ds.$$

Note that we take θ only in $[0, \pi)$ since the integral in (1) gives $\mathcal{R}f(t, \theta + k\pi) = \mathcal{R}f((-1)^k t, \theta)$, and so it suffices to consider the Radon transform for $\theta \in [0, \pi)$.

The Radon transform as given in (1) is clearly well-defined for smooth functions f that decay rapidly at infinity. This allows us to extend its definition by duality to the space of distributions, and in particular, to functions in $L^2(\mathbb{R}^2)$. Extensions to other spaces are possible using these arguments.

As an example, consider

$$f(\mathbf{x}) = \begin{cases} 1 & \|\mathbf{x}\| \leq 1, \\ 0 & \text{otherwise.} \end{cases}$$

Then an easy calculation gives

$$\mathcal{R}f(t, \theta) = \begin{cases} 2\sqrt{1-t^2} & |t| \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

Notice that in this case $\mathcal{R}f(t, \theta)$ has no dependence on θ , which follows from the fact that f is radially symmetric (or **isotropic**).

The Radon transform formalizes the collection of line integrals we began to study in Lecture 8: For θ fixed, we can interpret the function $t \mapsto \mathcal{R}f(t, \theta)$ as the intensity of rays that traverse f across $L_{t, \theta}$. Consequently, the problem of recovering the absorption coefficient is equivalent to that of inverting the Radon transform.

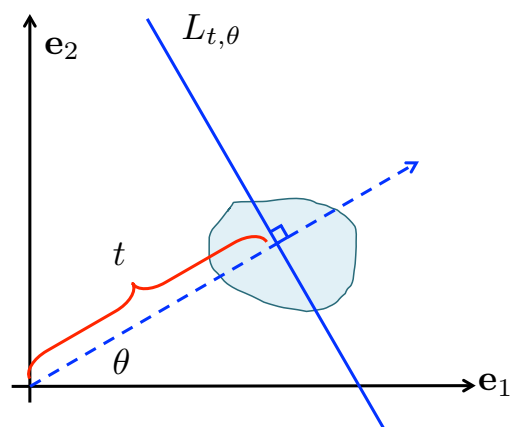


Figure 3: Parameterization of the lines along which the Radon transform is computed