

Lecture 10 — February 11, 2021

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1 Outline

Agenda: X-ray tomography

1. Backprojection
2. Radon inversion formula
3. Ill-posedness of the Radon inversion

Last Time: We introduced the topic of X-ray tomography. After suggesting a model that simplifies the relevant physics, we arrived at the problem of determining an object from its Radon transform; that is, from the collection of line integrals, each giving the projection of the object in a particular spatial direction. We asked first, is this possible? And second, if so, is there an inversion formula?

2 Radon transform

Recall the **Radon transform** of f as the function $\mathcal{R}f : \mathbb{R} \times [0, \pi) \mapsto \mathbb{C}$ given by

$$\mathcal{R}f(t, \theta) = \int_{L_{t, \theta}} f(\mathbf{x}) dS(\mathbf{x}) = \iint f(\mathbf{x}) \delta(\langle \mathbf{x}, \mathbf{n}_\theta \rangle - t) d\mathbf{x}, \quad (1)$$

where $\mathbf{n}_\theta = (\cos \theta, \sin \theta)$ is the unit vector with angle θ . $\mathcal{R}f(t, \theta)$ is the integral of f along the line $L_{t, \theta}$ whose direction is perpendicular to $\mathbf{n}_\theta$ and whose distance from the origin is t :

$$L_{t, \theta} = \{\mathbf{x} : \langle \mathbf{x}, \mathbf{n}_\theta \rangle = t\}.$$

3 The backprojection formula

We start our attempt to invert the Radon transform by discussing backprojection. Intuitively, the value $f(\mathbf{x})$ contributes to $\mathcal{R}f(t, \theta)$ whenever $\mathbf{x} \in L_{t, \theta}$. In the absence of any additional information, we will assume $f(\mathbf{x})$ contributes equally in all cases. Therefore, to compute the value of $f(\mathbf{x})$ we could try to average the values of $\mathcal{R}(t, \theta)$ along all lines that contain $\mathbf{x}$. By definition, $\mathbf{x} \in L_{t, \theta}$ if

and only if $t = \langle \mathbf{x}, \mathbf{n}_\theta \rangle$, and so the lines containing $\mathbf{x}$ are parameterized by θ . We thus define the **backprojection** of f by

$$\tilde{f}(\mathbf{x}) = \frac{1}{\pi} \int_0^\pi \mathcal{R}f(\langle \mathbf{x}, \mathbf{n}_\theta \rangle, \theta) d\theta.$$

However, $\tilde{f}$ is not equal to f in general, meaning this is *not* an inversion formula. In the case of the indicator function of the unit disk (see Fig. 1a), we have

$$\tilde{f}(\mathbf{x}) = \frac{2}{\pi} \int_0^\pi \sqrt{1 - |\langle \mathbf{x}, \mathbf{n}_\theta \rangle|^2} d\theta,$$

In particular, while f is compactly supported, $\tilde{f}$ vanishes nowhere (see Fig. 1b). Nevertheless, the backprojection of the unit disk closely resembles the unit disk; $\tilde{f}$ appears to be a blurred version of f . This gives us hope that some additional operations might be performed to obtain a true inversion formula.

4 The inverse Radon transform

To deduce an inverse for the Radon transform, we will make use of its connection to the Fourier transform. This result is called the **projection-slice theorem**.

Theorem 1 (projection-slice). *For any $\theta \in [0, \pi)$, the Fourier transform of the projection $\mathcal{R}f(\cdot, \theta)$ satisfies*

$$\int \mathcal{R}f(t, \theta) e^{-i\omega t} dt = \hat{f}(\omega \cos \theta, \omega \sin \theta).$$

In other words, measuring the Radon transform is equivalent to acquiring the Fourier transform of f along radial lines.

Proof. By direct calculation:

$$\int \mathcal{R}f(t, \theta) e^{-i\omega t} dt = \int \iint f(\mathbf{x}) \delta(\langle \mathbf{x}, \mathbf{n}_\theta \rangle - t) e^{-i\omega t} d\mathbf{x} dt = \iint f(\mathbf{x}) e^{-i\omega \langle \mathbf{x}, \mathbf{n}_\theta \rangle} d\mathbf{x} = \hat{f}(\omega \mathbf{n}_\theta).$$

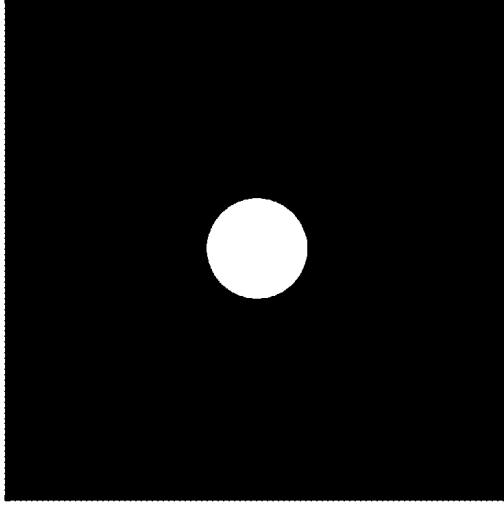
□

Consequently, we can recover f from its Radon transform using the fact we know $\hat{f}$ along radial lines. This leads to an expression similar to the backprojection formula, suitably called **filtered backprojection** or the **Radon inversion formula**.

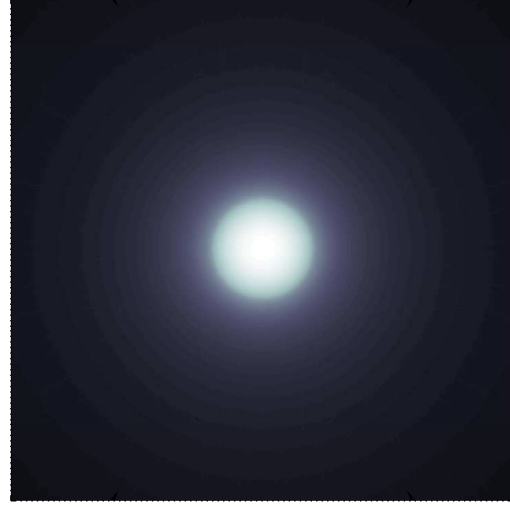
Theorem 2 (filtered backprojection). *We have*

$$f(\mathbf{x}) = \frac{1}{2\pi} \int_0^\pi (\mathcal{R}f(\cdot, \theta) * h)(\langle \mathbf{x}, \mathbf{n}_\theta \rangle) d\theta,$$

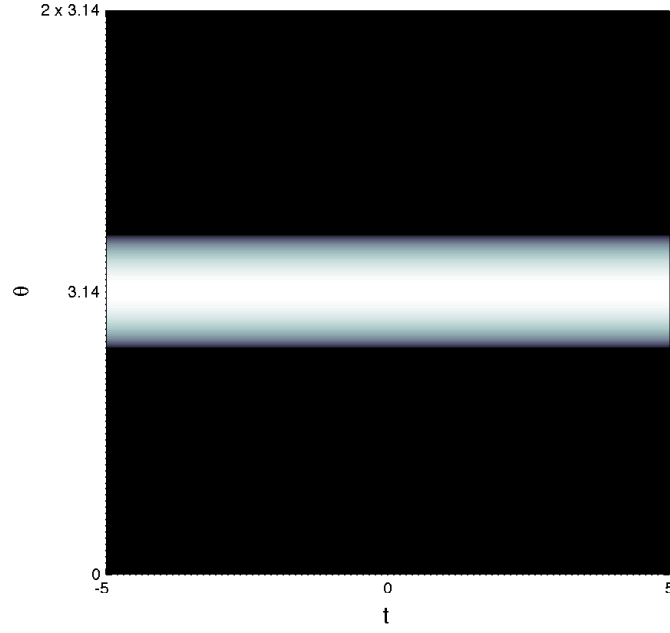
where h is such that $\hat{h}(\omega) = |\omega|$.



(a) $f(x) = \mathbb{I}\{\|x\| \leq 1\}$



(b) $\mathcal{R}^*\mathcal{R}f(x)$



(c) $\mathcal{R}f(t, \theta) = 2\mathbb{I}\{t \leq 1\}\sqrt{1-t^2}$

Figure 1: (a) Indicator function of the disk; (b) Backprojection reconstruction; (c) Radon transform of the indicator function of the disk.

The designation “filtered” comes from the fact that this inversion formula has the same form as the backprojection formula, but instead of directly averaging the contributions as described in the previous section, we first filter the Radon transform along the radial variable with the convolution kernel h .

Proof. Using the inverse Fourier transform, we have

$$\begin{aligned}
f(\mathbf{x}) &= \frac{1}{(2\pi)^2} \iint \hat{f}(\boldsymbol{\omega}) e^{i\langle \boldsymbol{\omega}, \mathbf{x} \rangle} d\boldsymbol{\omega} \\
&= \frac{1}{(2\pi)^2} \int_0^{2\pi} \int_0^\infty \hat{f}(r\mathbf{n}_\theta) e^{ir\langle \mathbf{n}_\theta, \mathbf{x} \rangle} r dr d\theta \\
&= \frac{1}{(2\pi)^2} \int_0^\pi \int_0^\infty \hat{f}(r\mathbf{n}_\theta) e^{ir\langle \mathbf{n}_\theta, \mathbf{x} \rangle} r dr d\theta \\
&\quad + \frac{1}{(2\pi)^2} \int_\pi^{2\pi} \int_0^\infty \hat{f}(r\mathbf{n}_\theta) e^{ir\langle \mathbf{n}_\theta, \mathbf{x} \rangle} r dr d\theta,
\end{aligned}$$

where we have changed to polar coordinates. The second integral can be rewritten as

$$\begin{aligned}
\int_\pi^{2\pi} \int_0^\infty \hat{f}(r\mathbf{n}_\theta) e^{ir\langle \mathbf{n}_\theta, \mathbf{x} \rangle} r dr d\theta &= - \int_\pi^{2\pi} \int_0^\infty \hat{f}(-r\mathbf{n}_\theta) e^{-ir\langle \mathbf{n}_\theta, \mathbf{x} \rangle} (-r) dr d\theta \\
&= \int_\pi^{2\pi} \int_{-\infty}^0 \hat{f}(r\mathbf{n}_{\theta-\pi}) e^{ir\langle \mathbf{n}_{\theta-\pi}, \mathbf{x} \rangle} (-r) dr d\theta \\
&= \int_0^\pi \int_{-\infty}^0 \hat{f}(r\mathbf{n}_\theta) e^{ir\langle \mathbf{n}_\theta, \mathbf{x} \rangle} (-r) dr d\theta,
\end{aligned}$$

from which it follows that

$$f(\mathbf{x}) = \frac{1}{(2\pi)^2} \int_0^\pi \int_{-\infty}^\infty \hat{f}(r\mathbf{n}_\theta) e^{ir\langle \mathbf{n}_\theta, \mathbf{x} \rangle} |r| dr d\theta = \frac{1}{2\pi} \int_0^\pi (\mathcal{R}f(\cdot, \theta) * h)(\langle \mathbf{n}_\theta, \mathbf{x} \rangle) d\theta.$$

The last equality is justified by the convolution theorem with $\hat{h}(r) = |r|$, applied alongside the projection-slice theorem. $\square$

In practice one cannot acquire information coming from rays in all of the continuum possible directions, and so one has to discretize in both t and θ . By the projection-slice theorem, this is equivalent to acquiring only a fraction of the radial lines in the Fourier domain, and therefore one has only partial information about the Fourier transform of f . A straightforward approach is to sum the corresponding filtered backprojections

$$f_{\text{rec}}(\mathbf{x}) \approx \sum_k (\mathcal{R}f(\cdot, \theta_k) * h)(\langle \mathbf{x}, \mathbf{n}_{\theta_k} \rangle),$$

where the convolution is also computed discretely.

5 Ill-posedness of the inverse problem

The inverse transform thus filters the Radon transform by h , which is declared to have Fourier transform $\widehat{h}(\omega) = |\omega|$, and then performs a scaled backprojection. By analogy with differentiation, for which $\frac{d}{dt}f(\omega) = i\omega\hat{f}(\omega)$, we see that the filter performs an operation similar to a derivative. Roughly speaking, then, the filter makes objects *more* singular. Another perspective is that while the backprojection formula produces images that are blurred, meaning edges are smoothed, convolving with h somehow recreates the edges from smooth data. Indeed, we are making objects more singular.

This feature of Radon inversion raises serious obstacles in its implementation. Consider the unavoidable fact that real-world measurements of a signal $\mathcal{R}f$ will contain noise. And any realization of noise (white, Brownian, grey, pink, etc.) is non-differentiable, meaning a filter that essentially takes derivatives will return a signal that has infinite energy! In a very dramatic way, the Radon inversion *amplifies* noise.

This complication is a result a mathematical observation that the inverse Radon transform

$$\mathcal{R}^{-1} : L^2([0, \pi) \times \mathbb{R}) \rightarrow L^2(\mathbb{R}^2)$$

is an unbounded operator, a fact we will see later. Consequently, two images that are far apart may have Radon transforms that are very close. In the presence of even a trivial amount of noise, there is a tremendous amount of ambiguity as to what the actual image is. In fact, it is possible to construct an analytic function whose integral along *any* line vanishes, and therefore its Radon transform is identically zero, yet the function itself is nonzero. Moreover, one can construct a sequence of functions diverging everywhere but whose Radon transforms converge uniformly (see [1]).

For these reasons, we see that the Radon inverse problem is **ill-posed**. Roughly speaking, an inverse problem is ill-posed when the solution is discontinuous (i.e. highly sensitive) with respect to the final conditions (the observed data). Ill-posed problems arise in many applications, and Radon inversion is just one example. The survey paper [2] provides a wealth of other examples. In the next section we discuss a prototypical case of ill-posedness: deconvolution.

References

- [1] Alexander Hertle. On the problem of well-posedness for the radon transform. In *Mathematical Aspects of Computerized Tomography*, pages 36–44. Springer, 1981.
- [2] SI Kabanikhin. Definitions and examples of inverse and ill-posed problems. *Journal of Inverse and Ill-Posed Problems*, 16(4):317–357, 2008.